“A Survey on the Four Families of Performance Measures”

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Agenda

- Motivation
- Literature
- The Four Families of Performance Measures
- Preliminary Conclusions
- Extensions
Motivation

- More than 100 ways to measure portfolio performance...
- Definition of the four main families of measures
- General expressions for each family
- Identification and explicit formulations for the more representative measures (and their close variants in Appendix)
- Intuition for each measure with some short criticisms
- Codes (MatLab, R,...) available soon on: www.performance-metrics.eu
Main and Recent Surveys about Performance Measurement


Some Books about Performance Measurement


The Four Representative Performance Measures


The Four Families of Performance Measures (1)

**General Form of Relative Performance Measures**

(most often expressed in return *per* unit of risk)

\[
PM_p = \mathcal{P}(\hat{r}_p) \times \left[ \mathcal{R}(\hat{r}_p) \right]^{-1},
\]

where \( \hat{r}_p \) are the (rescaled) returns, \( \mathcal{P}(\cdot) \) is a function that depends upon the observed performance, and \( \mathcal{R}(\cdot) \) is a corrected risk measure of the investor’s portfolio under study, such as \( \widehat{\mathcal{R}}(\cdot) = \mathcal{R}(\cdot) \times c(\cdot) \).

**Objective:** expressing the observed (rescaled) performance of the managed portfolio *per* unit of risk
The Four Families of Performance Measures (1)

The Reward-to-variability ratio (Sharpe, 1966)

\[ S_p = \left[ E(r_p) - r_f \right] \times \left( \sigma_{r_p} \right)^{-1}, \]

where \( E(\cdot) \) is the expectation operator, \( r_p \) and \( \sigma_{r_p} \) are respectively the returns and the total risk of the portfolio \( p \), and \( r_f \) is the risk-free asset.

Interpretation: this measure evaluates the compensation earned by the portfolio manager, as gauged by the expected excess return \( per \) unit of portfolio total risk.
# The Four Families of Performance Measures (1)

## The Main Sharpe-based Performance Measures**

<table>
<thead>
<tr>
<th>Authors</th>
<th>Name</th>
<th>Rescaled Performance ( \mathcal{P}(\hat{r}_p) )</th>
<th>Risk Measure ( \mathcal{R}(\hat{r}_p) )</th>
<th>Correction Coefficient ( c(\hat{r}_p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Morey and Vinod (2001)</td>
<td>Double Sharpe Ratio</td>
<td>( E(\hat{r}_p) - r_f )</td>
<td>Standard Deviation</td>
<td>( (\sigma_p)^{-1} )</td>
</tr>
<tr>
<td>Zakamouline and Koekbakker (2009)</td>
<td>Adjusted for Skewness Sharpe Ratio</td>
<td>( E(\hat{r}_p) - r_f )</td>
<td>Standard Deviation</td>
<td>( 1 \times \left{ \left{1 + b_{3,p} \times [m^2(\hat{r}_p) \times 3^{-1}] \times \left{ [E(\hat{r}_p) - r_f] \times \sigma_p^{-1}\right}^{-1}\right} \right}^{-1} )</td>
</tr>
<tr>
<td>Israælsen (2005)</td>
<td>Reward-to-absolute-excess return Ratio</td>
<td>( E(\hat{r}_p) - r_f )</td>
<td>Standard Deviation</td>
<td>( 1^{sgn[E(\hat{r}_p) - r_f] \times 1} )</td>
</tr>
<tr>
<td>Sharpe (1994)</td>
<td>Information Ratio</td>
<td>( E(\hat{r}_p) - r_f )</td>
<td>Tracking Error</td>
<td>-</td>
</tr>
<tr>
<td>Treynor (1965)</td>
<td>Reward-to-volatility Ratio</td>
<td>( E(\hat{r}_p) - r_f )</td>
<td>( Beta )</td>
<td>-</td>
</tr>
</tbody>
</table>

**where \( b_{3,p} \) corresponds to the investor’s relative preferences for skewness, \( m^3(\hat{r}_p) \) is the skewness of the underlying return distribution and \( sgn(\cdot) \) is the sign function.
## The Four Families of Performance Measures (1)

### Performance Measures based on Other Risk Measures

<table>
<thead>
<tr>
<th>Authors</th>
<th>Name</th>
<th>Rescaled Performance $\mathcal{P}(r_p)$</th>
<th>Corrected Risk Measure $\mathcal{R}(r_p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Darolles et al. (2009)</td>
<td>L-performance</td>
<td>TL-moment of order 1</td>
<td>TL-moment of order 2</td>
</tr>
<tr>
<td>Konno and Yamazaki (1991)</td>
<td>Mean Absolute Deviation Ratio</td>
<td>$E(r_p) - r_f$</td>
<td>Mean Absolute Deviation</td>
</tr>
<tr>
<td>Caporin and Lisi (2011)</td>
<td>Reward-to-range Ratio</td>
<td>$E(r_p) - r_f$</td>
<td>Range</td>
</tr>
<tr>
<td>Young (1998)</td>
<td>Minimax Ratio</td>
<td>$E(r_p) - r_f$</td>
<td>Minimax Portfolio</td>
</tr>
<tr>
<td>Dowd (2000)*</td>
<td>Reward-to Value-at-Risk Ratio</td>
<td>$E(r_p) - r_f$</td>
<td>Value-at-Risk</td>
</tr>
<tr>
<td>Sortino and Satchell (2001)</td>
<td>Reward-to $\sigma$-Lower Partial Moments Ratio</td>
<td>$E(r_p) - r_f$</td>
<td>Lower Partial Moment of order $\sigma$</td>
</tr>
<tr>
<td>Martin and McCann (1989)**</td>
<td>Ulcer Performance Index</td>
<td>$E(r_p) - r_f$</td>
<td>Average Squared Weekly Drawdowns</td>
</tr>
</tbody>
</table>

*See also Favre and Galeano (2002) who use the modified Value-at-Risk and Martin et al. (2003) who refers to the Conditional Value-at-Risk.
**See also Burke (1994), Young (1991) and Kestner (1996) which respectively based their measures on the Total Squared Monthly Drawdowns, the Maximum Drawdown and the Average Yearly Maximum Drawdowns.
The Four Families of Performance Measures (1)

Some Main Limits of Relative Performance Measures

- Rankings are consistent only if portfolio returns are elliptically distributed and/or the representative agent has a quadratic utility function.

- Use of derivative instruments (fat-tailed and skewed return distributions) or/and strategies with (time-varying) leverage effects yield to misleading conclusions (see Kao, 2002; Amin and Kat, 2003a and 2003b; Gregoriou and Gueyie, 2003)

*except for Zakamouline and Koekebakker (2009).
General Form of Absolute Performance Measures (expressed in return)

\[ PM_p = \Pi \left[ P(r_p), P^{th}(r_p | \Omega) \right], \]

where \( \Pi(\cdot;\cdot) \) is a transformation function (generally linear), \( \widehat{r}_p \) are the (rescaled) returns, \( P(\cdot) \) is a function that depends upon the observed performance and \( P^{th}(\cdot) \) is a function that is related to the theoretical performance of a model reference portfolio, conditionally to a set of information denoted \( \Omega \).

Objective: comparing the observed (rescaled) performance of the managed portfolio to its theoretical performance, considering a model
The Four Families of Performance Measures (2)

The \textit{Alpha} (Jensen, 1968)

\[
\alpha_p^J = \left[ E(r_p) - r_f \right] - \left[ E(r_m) - r_f \right] \times \beta_{p,m},
\]

where \( E(\cdot) \) is the expectation operator, \( r_p \) are the returns of the portfolio \( p \), \( r_m \) are the returns of the market portfolio \( m \), \( r_f \) is the risk-free rate, \( \beta_{p,m} \) is the sensitivity of investor’s portfolio returns with respect to market portfolio returns.

\textbf{Interpretation}: this measure assesses the \textit{extra} performance, realized by the portfolio manager, given its sensitivity to systematic risk.
## The Four Families of Performance Measures (2)

### The Main Jensen-type Performance Measures*

<table>
<thead>
<tr>
<th>Authors</th>
<th>Name</th>
<th>Rescaled Performance $\mathcal{P}(r_p)$</th>
<th>Theoretical Performance $\mathcal{P}^\text{th}(r_p \mid \Omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fama (1972)</td>
<td>Net Selectivity Index</td>
<td>$E(r_p) - r_f$</td>
<td>$\left{ \left[ E(r_m) - r_f \right] \times \left( \sigma_m \right)^{-1} \right} \times \sigma_p$</td>
</tr>
<tr>
<td>Black (1972)</td>
<td>Zero-beta CAPM</td>
<td>$E(r_p) - E(r_z)$</td>
<td>$[E(r_m) - E(r_z)] \times \beta_{p,z}$</td>
</tr>
<tr>
<td>Treynor and Mazuy (1966)</td>
<td>Market Timing Model</td>
<td>$E(r_p) - r_f$</td>
<td>$\left[ E(r_m) - r_f \right] \times \beta_{1,p,m}$ + $\left[ E(r_m) - r_f \right]^2 \times \beta_{2,p,m}$</td>
</tr>
<tr>
<td>Henriksson and Merton (1981)</td>
<td>Parametric Market Timing Model</td>
<td>$E(r_p) - r_f$</td>
<td>$\left[ E(r_m) - r_f \right] \times \beta_{1,p,m}$ + $\max\left[ r_f - E(r_m), 0 \right] \times \beta_{2,p,m}$</td>
</tr>
<tr>
<td>Connor and Korajczyk (1986)</td>
<td>Multi-factor Model</td>
<td>$E(r_p) - r_f$</td>
<td>$\sum_{k=1}^{K} E(F_k) \times \beta_{p,F_k}$</td>
</tr>
<tr>
<td>Ferson and Schadt (1996)</td>
<td>Multi-factor Conditional Model</td>
<td>$E(r_p) - r_f$</td>
<td>$\sum_{k=1}^{K} E(F_k) \times \beta_{p,F_k}(t)$</td>
</tr>
</tbody>
</table>

*where $r_z$ are the returns of the zero-beta portfolio, $\beta_{1,p,m}$ and $\beta_{2,p,m}$ are the selectivity and the market timing coefficients, $F_k$ is the $k$-th loading factor and $\beta_{p,F_k}(t)$ is the sensitivity of the portfolio $p$ to $k$-th the loading factor at time $t$. 
The Four Families of Performance Measures (2)

Other Miscellaneous Absolute Performance Measures*

<table>
<thead>
<tr>
<th>Authors</th>
<th>Name</th>
<th>Performance Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Henriksson and Merton</td>
<td>Non-parametric Market Timing Model</td>
<td>$P_1(t) + P_2(t) - 1$</td>
</tr>
<tr>
<td>Moses * et al. (1987)</td>
<td>Diversification-adjusted Alpha Measure</td>
<td>$\alpha_p \times (\eta_{p,m})^{-1}$</td>
</tr>
<tr>
<td>Grinblatt and Titman</td>
<td>Positive Period Weighting Measure</td>
<td>$\sum_{t=1}^{T} w_t \times (r_{p,t} - r_f)$</td>
</tr>
<tr>
<td>Modigliani and Modigliani (1997)</td>
<td>Risk-Adjusted Performance Measure</td>
<td>$E(r_p) \times (1 + \lambda_{p,m}) - r_f \times \lambda_{p,m}$</td>
</tr>
<tr>
<td>Cantaluppi and Hug</td>
<td>Efficiency Ratio</td>
<td>$[E(r_p) - r_f] \times [E(r_B) - r_f]^{-1}$</td>
</tr>
<tr>
<td>Muralidhar (2001)</td>
<td>Correlation-Adjusted Performance</td>
<td>$[E(r_i) \times w_i] \times [E(r_B) \times w_B]$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ [r_f \times (1 - w_i - w_B)]$</td>
</tr>
<tr>
<td>Muralidhar (2002)</td>
<td>Skill, History And Risk-Adjusted Measure</td>
<td>${[E(r_i) \times w_i] \times [E(r_B) \times w_B] \times [r_f \times (1 - w_i - w_B)]}$</td>
</tr>
</tbody>
</table>

*where $P_1(t)$ and $P_2(t)$ are the probabilities (resp. up and down) of correct forecasts of the portfolio manager about market variations; $\eta_{p,m}$ evaluates the diversification premium earned by the manager, $\lambda_{p,m}$ corresponds to the diversification level of the investor's portfolio compared to that of the market portfolio, $C(S)$ depends on the correlation coefficient between the investor's portfolio and his benchmark and $w_i$ is linked to the length of observations.
The Four Families of Performance Measures (2)

The Main Limits of Absolute Performance Measures

- Performance computed with these measures are strongly influenced by the reference portfolio (market portfolio or proxy).

- Almost all of them assumes stability of the systematic risk sensitivities of the investor’s portfolio over time.

- Disregards skewness and kurtosis of the studied portfolio returns which may alter rankings when using investment strategies based on derivative instruments.

* except for Ferson and Schadt (1996).
** except for Hwang and Satchell (1998).
General Form of Density-based Performance Measures (expressed in scalar terms – no unit)

\[ PM_p = \mathcal{P}^+ \left( \hat{r}_p \right) \times \left[ \mathcal{P}^- \left( \hat{r}_p \right) \right]^{-1}, \]

where \( \hat{r}_p \) are the (rescaled) returns, \( \mathcal{P}^+ (\cdot) \) is a function that depends upon the observed performance and \( \mathcal{P}^- (\cdot) \) is a measure focusing on a specific (left) part of the support of the density of returns.

Objective: comparing the observed (rescaled) performance of the managed portfolio to an expression depending on its losses.
Most of the Density-based Performance Measures can be expressed such as:

\[ PM_p = \mathcal{H}(\tau_1, \tau_2, \tau_3, \tau_4, o_1, o_2, k_1, k_2) \]

\[ = \left[ (-1)^{o_1} ES_{\tau_3, (-x_{r_{p_i}}^{*}, \tau_1)}^{o_1} \right] (k_1)^{-1} \times \left\{ (-1)^{o_2} ES_{\tau_4, (x_{r_{p_i}}^{*}, \tau_2)}^{o_2} \right\} (k_2)^{-1} \]^{-1} ,

where \( x_{r_{p_i}}^{*} \) corresponds to the portfolio order statistics \( r_p \) in excess of a threshold \( \tau_i \) for computing gains (for \( i = 1 \)) and losses or risk (for \( i = 2 \)), \( \tau_3 \) is a threshold specifying the right part of the support of the return density (i.e. gains) and \( \tau_4 \) is another threshold associated with the left part (i.e. losses), \( o_1 \) and \( o_2 \) are intensification constants, \( k_1 \) and \( k_2 \) are normalizing constants.
The Four Families of Performance Measures (3)

The **Omega** measure (Keating and Shadwick, 2002)

\[
O_p = E\left[\left( r_p - \tau \right) | r_p > \tau \right] \times \left\{ -E\left[\left( r_p - \tau \right) | r_p < \tau \right] \right\}^{-1},
\]

where \(E(\cdot|\cdot)\) is the conditional expectation operator, \(r_p\) corresponds to the return on a portfolio \(p\) and \(\tau\) is a threshold.

**Interpretation:** it compares the potential gains of the managed portfolio over its potential losses, both defined according to a threshold
# The Four Families of Performance Measures (3)

## The Main Density-based Performance Measures

<table>
<thead>
<tr>
<th>Authors</th>
<th>Name</th>
<th>Rescaled Performance ( P^+ (\bar{r}_p) )</th>
<th>Downside Performance ( P^- (\bar{r}_p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bernardo and Ledoit (2000)</td>
<td>Gain-Loss ratio</td>
<td>( E[\max(\bar{r}_p - r_f, 0)] )</td>
<td>( E[\max(r_f - \bar{r}_p, 0)] )</td>
</tr>
<tr>
<td>Farinelli and Tibiletti (2008)</td>
<td>One-sided Risk Measure</td>
<td>( \left[HPM_{o1} (\bar{r}_p - r_f) \right]^{\frac{1}{o_1}} )</td>
<td>( \left[(-1)^{o_2} \times LPM_{o2} (\bar{r}_p - r_f) \right]^{\frac{1}{o_2}} )</td>
</tr>
<tr>
<td>Biglova et al. (2004)</td>
<td>Rachev ratio</td>
<td>( -E[(r_f - \bar{r}_p)(r_f - \bar{r}<em>p) &lt; VaR</em>{a1,p}] )</td>
<td>( -E[(\bar{r}_p - r_f)(\bar{r}<em>p - r_f) &lt; VaR</em>{a2,p}] )</td>
</tr>
<tr>
<td>Biglova et al. (2004)</td>
<td>Generalized Rachev ratio</td>
<td>( (-1)^{o_1} \times E[(r_f - \bar{r}_p)^{o_1}(r_f - \bar{r}<em>p) &lt; VaR</em>{a1,p}] )</td>
<td>( (-1)^{o_2} \times E[(\bar{r}_p - r_f)^{o_2}(\bar{r}<em>p - r_f) &lt; VaR</em>{a2,p}] )</td>
</tr>
</tbody>
</table>

*where \( r_f \) is the risk-free rate, \( MAR \) is the Minimum Acceptable Return, \( a_i \) with \( i = (1,2) \) corresponds to intensification constants.
The Main Limits of Density-based Performance Measures

- Performance measures are highly related to the threshold (risk free rate, Minimum Acceptable Return or Value-at-Risk)
- The link with the utility function of the studied agent is not straightforward
- Density are subject to model risk
The Four Families of Performance Measures (4)

General Form of Utility-based Performance Measures (expressed in return per unit of util)

\[
PM_p = \mathcal{G} \circ \left\{ E\left[ \mathcal{V}\left( \hat{r}_p \right) \right] \right\},
\]

where \( \hat{r}_p \) are the (rescaled) returns, \( E(\cdot) \) is the expectation operator, \( \mathcal{V}(\cdot) \) is a value (or utility) function and \( \mathcal{G}(\cdot) \) is a specific function that depends upon the performance of the investor’s portfolio.

Objective: evaluating the portfolio performance from explicit representative value (or utility) functions.
The Morningstar Risk-Adjusted Return (Morningstar, 2002)

\[ MRAR_p = E \left[ \left( 1 + r_p \right)^{-A} \right]^{\frac{12}{A}}, \]

where \( E(\cdot) \) is the expectation operator, \( r_p \) are the returns of a portfolio \( p \), \( A \) (with \( A \neq 0 \)) is the risk aversion coefficient of the studied investor.

**Interpretation:** incorporating the behavior of the agent, through a Power Utility Function, for assessing the portfolio performance, given a risk aversion coefficient
The Four Families of Performance Measures (4)

The Main Utility-based Performance Measures*

<table>
<thead>
<tr>
<th>Authors</th>
<th>Name</th>
<th>Transformed Performance ( \mathcal{G}(\cdot) )</th>
<th>Value (or utility) Function ( \mathcal{V}(\tilde{r}_p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stutzer (2000)</td>
<td>Performance Index</td>
<td>[ \arg\max_{\theta \in \mathbb{R}^+} {-\log[E(\cdot)]} ]</td>
<td>[ \exp[\theta \times (r_p - r_B)] ]</td>
</tr>
<tr>
<td>Kaplan (2005)</td>
<td>Lambda measure</td>
<td>[ \arg\max_{\theta \in \mathbb{R}^+} {E(\cdot)} ]</td>
<td>[ \theta \times (r_p - r_B) - \delta_p ]</td>
</tr>
<tr>
<td>Gemmill et al. (2006)</td>
<td>Loss-Averse Performance Measures</td>
<td>[ P(r_p \geq 0) \times E(U_1) \times [P(r_p &lt; 0) \times E(U_2)]^{-1} ]</td>
<td>[ U_1 = \max \left[ (r_p - r_B)^{k_1}, 0 \right] ] [ U_2 = \min \left[ (r_p - r_B)^{k_2}, 0 \right] ]</td>
</tr>
<tr>
<td>Ingersoll et al. (2007)</td>
<td>Manipulation-Proof Performance Measure</td>
<td>[ \frac{(1 - A)\Delta t}{1} \times \ln[E(\cdot)] ]</td>
<td>[ \left[ (1 + r_p) \times (1 + r_f)^{-1} \right]^{1-A} ]</td>
</tr>
</tbody>
</table>

*where \( \theta \) is a constant, \( \delta_p \) is a penalty function, \( k_1 \) and \( k_2 \) are constants, \( \Delta t \) is the frequency of observations and \( A \) is the risk aversion coefficient.
The Four Families of Performance Measures (4)

- The Main Limits of Utility-based Performance Measures

  - Strongly dependent on the utility function (exponential, power or logarithmic) characterizing the behavior of the studied investor

  - Highly sensitive to the investor’s attitude towards risk through the risk aversion coefficient (which can be time-varying)

  - Rankings obtained with these measures are directly related to the benchmark (risk free rate or proxy)
Preliminary Conclusions

We propose a Survey of performance measures in a uniform and comprehensive framework through four main families clearly identified, namely relative, absolute, density-based and utility-related performance measures.

We define these main families according to two essential criteria: the unit in which it is expressed, and the way the measure is built.

We formulate a general expression for each of the four categories of performance measures, in which most of the performance measures fits.
Extensions

On our on-going research agenda, we have planned to add new performance measures (30 extra or so), in order to complete our actual Survey.

Next step will be to perform some empirical applications in order to compare the properties of collected measures (rank correlation tests, persistence, lucky versus star funds studies, etc.).

We would like also to complement our intuitions about a new performance measure (called “Generalized Performance Measure”), which can be seen as a generalization of the four main categories presented in this paper.
We are grateful to Christophe Boucher and Patrick Kouontchou for help and encouragement in preparing this work.

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